PRIMITIVE ENRIQUES VARIETIES

j/w. Denisi - Ríos Ortiz - Xie and Denisi - Xie

$/\mathbb{C}$, mnfds are connected, varieties are normal and projective

MMP $\overset{\text{predicts}}{\rightsquigarrow}$ every proj. var. with mild sing. can be decomposed, up to br. equiv., into the following 3 fundamental building blocks:

Fano varieties ($-K$: ample)

Calabi-Yau varieties (K: trivial)

canonically polarized varieties (K: ample)

· <u>THM</u> (Beauville-Bogomolov decomp.):

X: compact Kähler mnfd with $c_1(X) = 0 \in H^2(X, \mathbb{R})$

$\Rightarrow \exists \tilde{X} \to X$: finite étale cover s.t.

$$\tilde{X} \cong T \times \prod Y_j \times \prod Z_k ,$$

where

- T: complex torus
- Y_j: irreducible Calabi-Yau mnfd
- Z_k: irreducible holomorphic symplectic (IHS) mnfds

· <u>DEF</u> (IHS): X: compact Kähler mnfd s.t. $\pi_1(X) = \{1\}$ and $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \sigma$, where σ is a symplectic form (i.e., closed and non-degenerate).

- $\dim_{\mathbb{C}} X = 2n$ (even)
- $K_X = \Omega_X^{2n} \cong \mathcal{O}_X$

- $\dim X = 2 \Rightarrow X$: K3 surface

- <u>DEF</u> (Enriques mnfd) : Y: cplx mnfd s.t. $\pi_1(Y) \neq \{1\}$ and $Y^{univ} = X$: IHS mnfd.

  $\rightsquigarrow |\pi_1(Y)| < +\infty \Rightarrow Y$: compact

  $\rightsquigarrow \dim_{\mathbb{C}} Y = 2n$ (even)

  $\rightsquigarrow \dim Y = 2 \Rightarrow Y$: Enriques surface.

  * All currently known examples of Enriques mnfds were constructed by Oguiso-Schröer and Boissière-Nieper-Wißkirchen-Sarti. ($d \in \{2, 3, 4\}$)

!<u>REM</u>: Y : Enr. mnfd, $\dim Y = 2n$, $X = Y^{univ}$.

(i) $\pi_1(Y)$: cyclic of order d (called the index of Y) and satisfies $d | n+1$.

(ii) $\pi_1(Y) \curvearrowright X$: freely ($\underset{HLFFF}{\overset{e.g.}{\Longleftarrow}}$ non-symplectically)

$\Rightarrow Y \cong X / \pi_1(Y)$.

(iii) Both X and Y are automatically projective.

$H^0(Y, \Omega_Y^2) = \{0\}$

(iv) $\omega_Y \in Pic(Y)$ is a line bundle of order d.

- <u>THM</u> (singular Beauville-Bogomolov decomp.):

  X : normal proj. var. with klt sing and $c_1 \equiv 0$

  $\Rightarrow \exists \tilde{X} \to X$: finite quasi-étale cover s.t.

$$\tilde{X} \cong A \times \prod Y_j \times \prod Z_k ,$$

where

· A : abelian variety,
· Y_j : irreducible Calabi-Yau variety,
· Z_k : irreducible symplectic variety.

$\rightsquigarrow$ irreducible Enriques variety

$*$: Issues : they do not behave well under deformations and operations of the MMP.

· **DEF** (PSV) : A primitive symplectic variety (PSV) is a proj. symplectic variety X (i.e., $\exists\, \sigma \in H^0(X, \Omega_X^{[2]})$: symplectic + X has rational singularities) s.t. $H^1(X, \mathcal{O}_X) = \{0\}$ and $H^0(X, \Omega_X^{[2]}) = \mathbb{C} \cdot \sigma$.

! **REM**:

1) smooth PSVs = (proj.) IHS mnfds [Schwald]
ISVs are PSVs (but the converse fails)

2) PSVs behave well under [Bakker-Lehn]
· deformations (every small locally trivial deformation of a PSV is again a PSV + $\exists$ local Torelli thm + ...)
· operations of the MMP
in particular, $\exists$ termination statement for IHS mnfds due to [Lehn-Pacienza].

· **DEF** (PEV) : A primitive Enriques variety is the quotient $Y = X/G$ of a primitive symplectic variety X by a finite grp $G \subseteq \mathrm{Aut}(X)$ whose action on X is non-symplectic and free in codimension one.

REM: $G \curvearrowright X$: non-symplectic

$\Rightarrow$ 1) G : cyclic, generated by a purely non-symplectic automorphism $g \in \mathrm{Aut}(X)$.

2) $\dim X \geq 4 \Rightarrow G \curvearrowright X$: free in codim one.

· **REM**: Y: PEV

1) Y: has klt singularities

2) Y: not uniruled iff Y has (at worst) canonical singularities

3) $K_Y \sim_{\mathbb{Q}} 0$ (but ~ 0 can happen, cf. smooth case)

4) $q(Y) = H^1(Y, \mathcal{O}_Y) = \{0\}$

$H^0(Y, \Omega_Y^{[2]}) = \{0\}$.

$\left(\begin{array}{l} \text{In general,} \\ H^0(Y, \Omega_Y^{[t]}) \cong H^0(X, \Omega_X^{[t]})^G \end{array} \right)$

5) Y: smooth PEV $\Rightarrow Y$: Enriques mnfd.

Proof: $\exists\, f : X \longrightarrow Y = X/G$: quasi-étale $\xRightarrow[\text{purity of the branch locus}]{Y:\text{ smooth}}$

$\Rightarrow f$: étale $\Rightarrow X$: smooth $\xRightarrow{X:\text{PSV}}$

$\Rightarrow X$: IHS mnfd $\Rightarrow \pi_1(X) = \{1\}$

$\Rightarrow f$: universal cover of $Y \xRightarrow{G \neq \{\mathrm{Id}_X\}}$

$\Rightarrow Y$: Enriques mnfd. ●

Slogan: PEVs behave well under deformations and operations of the MMP.

· **THM** (Denisi, Ríos Ortiz, T, Xie): Let (Y, B_Y) be a log canonical pair, where Y is an Enriques mnfd. Then any $(K_Y + B_Y)$-MMP terminates with a minimal model $(Y', B_{Y'})$ of (Y, B_Y), where

Y': $\mathbb{Q}$-factorial PEV with canonical singularities and $B_{Y'} \geq 0$: nef $\mathbb{Q}(\mathbb{R})$ divisor on Y'.

PROOF (Idea): Reduce to the termination statement of Lehn-Pacienza via a lifting construction. ∎

· THM (Denisi, T., Xie): Any small locally trivial deformation of a PEV is again a PEV. In particular, if Y is a PEV of the form:

* $Y = \mathrm{Hilb}^n(S)/G$
* $Y = \mathrm{Kum}^n(A)/G$
* $Y = M_S(v,H)/G$

then any small locally trivial deformation of Y is again of the same form.

⇝ The "in particular" part of the statement uses crucially a local Torelli thm for PEVs [Denisi, T., Xie]

EXAMPLE:

S: projective K3 surface
$\tau \in \mathrm{Aut}(S)$: fixed-point-free involution ($\Rightarrow \tau$: anti-symplectic)

$S^{[n]} := \mathrm{Hilb}^n(S)$: proj. IHS mnfd of dim $2n$ [Beauville]
$\tau^{[n]} \in \mathrm{Aut}(S^{[n]})$: anti-symplectic involution

Case A: $n \geq 3$: odd $\xRightarrow[\text{Schröer}]{\text{Oguiso}}$ $\langle \tau^{[n]} \rangle \cong \mathbb{Z}/2\mathbb{Z} \curvearrowright S^{[n]}$ freely

$\Rightarrow Y := S^{[n]}/\langle \tau^{[n]} \rangle$: Enriques mnfd of dim $2n$ and index 2

$\Rightarrow \pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$, $K_Y \not\sim 0$
$2K_Y \sim 0$.

Case B : $n \geq 2$: even $\Longrightarrow \mathrm{Fix}(\tau^{[n]}) \neq \emptyset$

$\xRightarrow{\text{Beauville}} \mathrm{Fix}(\tau^{[n]}) = \bigsqcup_{\text{finite}} \left(\begin{array}{l}\text{Lagrangian} \\ \text{submfds of } S^{[n]}\end{array}\right)$

$\Rightarrow \langle \tau^{[n]} \rangle \curvearrowright S^{[n]}$: freely in codimension one (and non-symplectically by construction)

$\Rightarrow Y := S^{[n]} / \langle \tau^{[n]} \rangle$: PEV of dim $2n$ with the following props. :

- Y : $\mathbb{Q}$-factorical
- Y : canonical singularities ($\Rightarrow Y$: not uniruled)
- $K_Y \sim 0$ (cf. smooth case)
- $\pi_1(Y) = \{1\}$ (cf. smooth case)

$*$ $n = 2 \xRightarrow{\text{[CGM19]}} Y$: admits a crepant resolution by an ICY 4 fold ($\neq$ EM)
(cf. symplectic case)

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